ALGEBRA.

129. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, O.

$$\begin{vmatrix} 1-\binom{m-1}{0} & -\binom{m-1}{1} & \cdots & \binom{m-1}{m-2} \\ 1 & 1- & \binom{m-2}{0} \cdots & \binom{m-2}{m-3} \\ 1 & 0 & 1 \cdots & \binom{m-3}{m-4} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 \quad 1 & \binom{1}{0} \\ 1 & 0 & 0 \ldots 0 & 1 \end{vmatrix} \quad e=\sum_{n=\infty}^{n=1}\frac{n^m}{n\,!}. \text{ where } \binom{m-2}{k}=\frac{(n-2)\ldots(m-k-1)}{k\,!}$$

Solution by the PROPOSER.

Let $f(x)=e^{e^x}=1+e^x+\frac{e^{2x}}{2\,!}+\frac{e^{3x}}{3\,!}+\frac{e^{4x}}{4\,!}+\frac{e^{5x}}{5\,!}+\ldots+\frac{e^{rx}}{r\,!}+\ldots.$

$$=1+(1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\frac{x^4}{4\,!}+\frac{x^5}{5\,!}+\ldots.)$$

$$+\frac{1}{2\,!}(1+2x+\frac{(2x)^2}{2\,!}+\frac{(2x)^3}{3\,!}+\frac{(2x)^4}{4\,!}+\frac{(2x)^5}{5\,!}+\ldots.)$$

$$+\frac{1}{3\,!}(1+3x+\frac{(3x)^2}{2\,!}+\frac{(3x)^3}{3\,!}+\ldots)$$

....

$$+\frac{1}{r\,!}(1+rx+\frac{(rx)^2}{2\,!}+\frac{(rx)^3}{3\,!}+\ldots.)$$

....

....

$$=e+\sum_{r=1}^{r=\infty}a_r x^r, \text{ where } a_r=\frac{1}{r\,!}\sum_{k=1}^{k=\infty}\frac{k^r}{k\,!}.$$

But $f(x)=f(0)+\frac{f'(0)}{1\,!}x+\frac{f''(0)}{2\,!}x^2+\ldots.$

$\therefore f^m(0)=\sum_{k=1}^{k=\infty}\frac{k^m}{k\,!}\ldots.(1).$

Let $y=e^{e^x}$, then $\log y=e^x$.

$\therefore \dfrac{dy}{dx}=ye^x$. And by Leibnitz's Formula,

$$f^m(0)=f^{m-1}(0)+(m-1)f^{m-2}(0)+\frac{(m-1)(m-2)}{2\,!}f^{m-3}(0)+\dots\dots$$

$$+(m-1)f'(0)+f(0)=\sum_{k=1}^{k=\infty}\frac{k^m}{k\,!}.\ [\text{by } (1)] \qquad .(2).$$

Or, letting x_m represent $f^m(0)$, etc.,

$$x_m-x_{m-1}-C_{m-1,1}x_{m-2}-\dots\dots-C_{m-1,1}x_1=f(0)=e.$$

$$x_{m-1}-x_{m-2}-C_{m-2,1}x_{m-3}\dots\dots-C_{m-2,1}x_1=e$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$x_2-x_1=e$$

$$x_1=e$$

$$\therefore x_m=\begin{vmatrix} e & -\binom{m-1}{0} & -\binom{m-1}{1} & & \dots\dots\dots & -\binom{m-1}{m-2} \\ e & 1 & -\binom{m-2}{0} & & \dots\dots\dots & -\binom{m-2}{m-3} \\ e & 0 & 1 & & \dots\dots\dots & -\binom{m-3}{m-4} \\ e & 0 & 0 & 1 & \dots\dots & -\binom{m-4}{m-5} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ e & 0 & 0 & 0 & \dots\ 1 & -\binom{1}{0} \\ e & 0 & 0 & 0 & \dots\ 0 & 1 \end{vmatrix}$$

$$\therefore \begin{vmatrix} 1 & -\binom{m-1}{0} & -\binom{m-1}{1} & & \dots\dots\dots & -\binom{m-1}{m-2} \\ 1 & 1 & -\binom{m-2}{0} & & \dots\dots\dots & -\binom{m-2}{m-3} \\ 1 & 0 & 1 & & \dots\dots\dots & -\binom{m-3}{m-4} \\ 1 & 0 & 0 & 1 & \dots\dots & -\binom{m-4}{m-5} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 & \dots\dots 1 & -\binom{1}{0} \\ 1 & 0 & 0 & 0 & \dots\dots 0 & 1 \end{vmatrix} e=\sum_{k=1}^{k=\infty}\frac{k^m}{k\,!}\dots\dots(3).$$

By using (2) or (3) the following special relations have been obtained :

$$2e=\sum_{k=1}^{k=\infty}\frac{k^2}{k!},\ 5e=\sum_{k=1}^{k=\infty}\frac{k^3}{k!},\ 15e=\sum_{k=1}^{k=\infty}\frac{k^4}{k!},\ 52e=\sum_{k=1}^{k=\infty}\frac{k^5}{k!},\ 203e=\sum_{k=1}^{k=\infty}\frac{k^6}{k!},\ 877e=\sum_{k=1}^{k=\infty}\frac{k^7}{k!},$$

$$4140e=\sum_{k=1}^{k=\infty}\frac{k^8}{k!},\ 21147e=\sum_{k=1}^{k=\infty}\frac{k^9}{k!},\ 115975e=\sum_{k=1}^{k=\infty}\frac{k^{10}}{k!}.$$

Also solved by *G. B. M. ZERR*.

GEOMETRY.

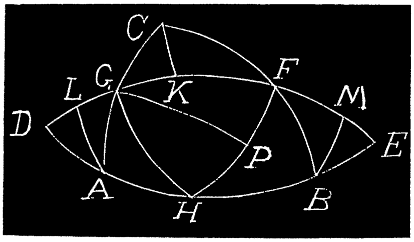
160. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let GFH be the spherical triangle formed by joining the mid-points of the sides of the spherical triangle ABC; E the spherical excess of ABC; β, p the base and altitude of GFH. Prove $\sin\frac{1}{2}E=\sin\beta\sin p$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Let $BA=c$, $GF=\gamma$, $FH=\beta$, $GH=\delta$, $GP=p$.

Draw AL, BM, CK perpendicular to $DGFC$. Now $DABE=DGFE=\pi$.



$AD=BE=\frac{1}{2}(\pi-c)$, $DL=ME$, $LG=CK$, $KF=FM$.

$\therefore 2(DL+GK+KF)=\pi$. Also $2GK+2KF=2\gamma$.

$\therefore 2DL+2\gamma=\pi$, or $DL=\frac{1}{2}\pi-\gamma$.

$\angle DAL=\angle EBM$, $\angle LAG=\angle GCK$, $\angle KCF=\angle FBM$.

$\therefore 2\angle DAL+C+A+B=2\pi$.

$\therefore \angle DAL=\pi-\frac{1}{2}(A+B+C)=\pi-s=\frac{1}{2}(\pi-E)$.

$\cos DAL=\sin D\cos DL$. $\cos\frac{1}{2}(\pi-E)=\sin D\cos(\frac{1}{2}\pi-\gamma)$.

$\therefore \sin\frac{1}{2}E=\sin D\sin\gamma$.

Now $\sin D:\sin DFH=\sin\beta:\sin DH$. But $DH=\frac{1}{2}\pi$.

$\therefore \sin D=\sin\beta\sin DFH$. But $\sin p=\sin\gamma\sin DFH$.

$\therefore \sin D=(\sin\beta\sin p)/\sin\gamma$.

$\therefore \sin\frac{1}{2}E=\sin\beta\sin p$.

Also solved by *J. SCHEFFER* and *L. C. WALKER*.

161. Proposed by MARCUS BAKER, U. S. Coast and Geodetic Survey Office, Washington, D. C.

A circle, radius r, is inscribed in a triangle ABC. In the angles A, B, and C are inscribed circles each touching two sides and the inscribed circle. There are six such circles. The first group of three have their centers between the in-centers and the vertices, and the second group of three does not. Let r_a, r_b, r_c denote the radii of the first group. Then this well known relation holds: $r=\sqrt{(r_ar_b)}+\sqrt{(r_br_c)}+\sqrt{(r_cr_a)}$. Let R_a, R_b, R_c denote the radii of the second group. Then this relation holds: